

Do all your writing on these six pages, using the backs if necessary. You must show your work to receive credit.

1. (10 points) Find an equation of the line through $(2, -3)$ parallel to the line $3x + 5y = 6$.

Slope: $m = \text{Same slope as } 3x + 5y = 6 \rightarrow m = -\frac{3}{5}$

Point: $(2, -3)$

Equation: $y + 3 = -\frac{3}{5}(x - 2)$

2. (10 points) Find an equation of the tangent line to the graph of $f(x) = x - x^{3/2} + 1$ at $x = 9$.

Need a) point $x = 9, y = f(9) = 9 - 9^{3/2} + 1 = -17$
 $(9, -17)$

b) slope $m = f'(9)$
 $f'(x) = 1 - \frac{3}{2}x^{1/2} \quad f'(9) = 1 - \frac{3}{2}\sqrt{9} = 1 - \frac{9}{2} = -\frac{7}{2}$

Equation: $y + 17 = -\frac{7}{2}(x - 9)$

3. (10 points) Let $f(x) = 3x^2 - 2x - 1$. Use the limit definition of the derivative to show that $f'(x) = 6x - 2$.

Def $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

So $f'(x) = \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 2(x+h) - 1 - [3x^2 - 2x - 1]}{h}$
 $= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 1 - 3x^2 + 2x + 1}{h}$
 $= \lim_{h \rightarrow 0} \frac{6xh + 3h^2 - 2h}{h}$
 $= \lim_{h \rightarrow 0} 6x + 3h - 2 = \underline{\underline{6x - 2}}$

#4. (10 points) Complete each of the following differentiation formulae:

(a) $\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$ Chain rule

(b) $\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ quotient rule

#5. (10 points) Solve each equation for x:

(a) $7 = 1 + 18 \ln(x-2)$

$6 \quad 18 \ln(x-2)$

$\frac{1}{3} = \ln(x-2)$

(b) $3 + e^{-7x} = 25$

$e^{-7x} = 22$

$-7x = \ln(22)$

$e^{\frac{1}{3}} = x-2$
 $x = e^{\frac{1}{3}} + 2$

$\ln 22$
 $x = \frac{\ln 22}{-7}$

#6. (30 points) Find the derivative of each of the following. You need not simplify.

(a) $f(x) = 10x^8 - 8x^3 + 2\sqrt{x} - 3$

$f'(x) = 80x^7 - 24x^2 + 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} = 80x^7 - 24x^2 + \frac{1}{\sqrt{x}}$

(b) $y = \frac{1}{\sqrt{3x^2+5}} = (3x^2+5)^{-\frac{1}{2}}$

$y' = -\frac{1}{2} (3x^2+5)^{-\frac{3}{2}} \cdot (6x)$

(c) $y = \frac{x^2+1}{2x^3-4x+3}$

$y' = \frac{(2x^3-4x+3)(2x) - (x^2+1)(6x^2-4)}{(2x^3-4x+3)^2}$

(d) $y = x^4 \ln x$

$y' = 4x^3 \ln x + x^4 \cdot \frac{1}{x}$

(e) $g(x) = e^{x^2+2x}$

$g'(x) = (2x+2) \cdot e^{x^2+2x}$

(f) $h(x) = \ln(7x^3+2x^2+5)$

$h'(x) = \frac{1}{7x^3+2x^2+5} (21x^2+4x)$

#7. (15 points) Use calculus to find all relative maximum and relative minimum points of $y = f(x) = \frac{-2}{3}x^3 + 3x^2 - 5$. Use this information to sketch the graph.

y-intercept = -5

$$f'(x) = -2x^2 + 6x$$

$$-2x^2 + 6x \stackrel{\text{set}}{=} 0$$

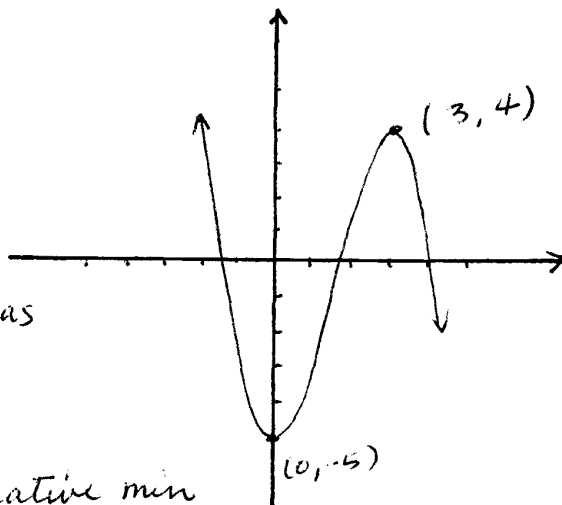
$$-2x(x-3) = 0$$

$x=0, x=3$ are possible extremas

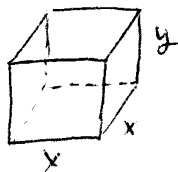
$$f''(x) = -4x + 6$$

$$\left. \begin{array}{l} f''(0) = 6 > 0 \\ f(0) = -5 \end{array} \right\} \text{so } (0, -5) \text{ is a relative min}$$

$$\left. \begin{array}{l} f''(3) = -6 < 0 \\ f(3) = 4 \end{array} \right\} \text{so } (3, 4) \text{ is a relative max}$$



#8. (15 points) A box with a square base is to be built to have a volume of 80 cubic feet. The material for the square top and bottom of the box costs \$2.50 per square foot while the material for the other four sides of the box costs \$2.00 per square foot. Use calculus to find the dimensions of the box that minimize the cost of construction.



$$\text{Cost } (C) = 2.5 \cdot (2) x^2 + 2 \cdot (4) \cdot x y$$

$$C = 5x^2 + 8xy$$

$$\rightarrow y = \frac{80}{x^2}$$

$$C(x) = 5x^2 + 8x \left(\frac{80}{x^2} \right)$$

$$C(x) = 5x^2 + \frac{640}{x}$$

$$C'(x) = 10x - \frac{640}{x^2} \stackrel{\text{set}}{=} 0$$

$$10x = \frac{640}{x^2}$$

$$10x^3 = 640$$

$$x^3 = 64 \quad \text{so } x = \sqrt[3]{64} = 4$$

$$\text{so, } x = 4$$

$$y = \frac{80}{4^2} = 5$$

$$C''(x) = 10 + \frac{1280}{x^3}$$

$$C''(4) = 10 + 20 = 30 > 0$$

So $x=4$ is a min

∴ Dimension is :

$$\underline{4 \times 4 \times 5 \text{ feet}}$$

Given:

$$V = x^2 y = 80 \text{ ft}^3$$

- # 9. (10 points) The revenue for a company's commodity is $300x - 2x^2$ where x is the number of units produced. The cost of producing x units is $20 + 60x - x^2$. What number of units produced results in the maximum profit?

$$\text{Profit} = \text{Revenue} - \text{cost}$$

$$P(x) = R(x) - C(x)$$

$$P(x) = 300x - 2x^2 - [20 + 60x - x^2]$$

$$P(x) = 300x - 2x^2 - 20 - 60x + x^2$$

$$P(x) = -x^2 + 240x - 20$$

$$P'(x) = -2x + 240 \stackrel{\text{set}}{=} 0$$

$$x = 120$$

$$P''(x) = -2 < 0$$

$\therefore x = 120$ units will maximize profit

- # 10. (10 points) The function $f(x) = xe^x + e^x$ has one relative minimum and no other extreme values. Find the value of x at which the minimum occurs.

Find x where $f'(x) = 0$

$$f'(x) = e^x + xe^x + e^x$$

$$f'(x) = 2e^x + xe^x$$

$$f'(x) = e^x(2+x) \stackrel{\text{set}}{=} 0$$

$$e^x \neq 0 \text{ so, } 2+x = 0$$

$$x = -2$$

- # 11. (10 points) The world rhinoceros population decreases at a rate proportional to its size. In 1950 there were 20,000 rhinoceroses, while in 1980 there were only 5,000. How many will there be in the year 2000? Give a decimal answer.

$$P(t) = P_0 e^{kt} \quad P_0 = 20,000 \quad \text{and} \quad P(30) = 5,000$$

$$P(30) = 20,000 e^{30k} = 5,000$$

$$e^{30k} = \frac{5000}{20000} = \frac{1}{4}$$

$$30k = \ln\left(\frac{1}{4}\right)$$

$$k = \frac{\ln\left(\frac{1}{4}\right)}{30} \approx -0.04621$$

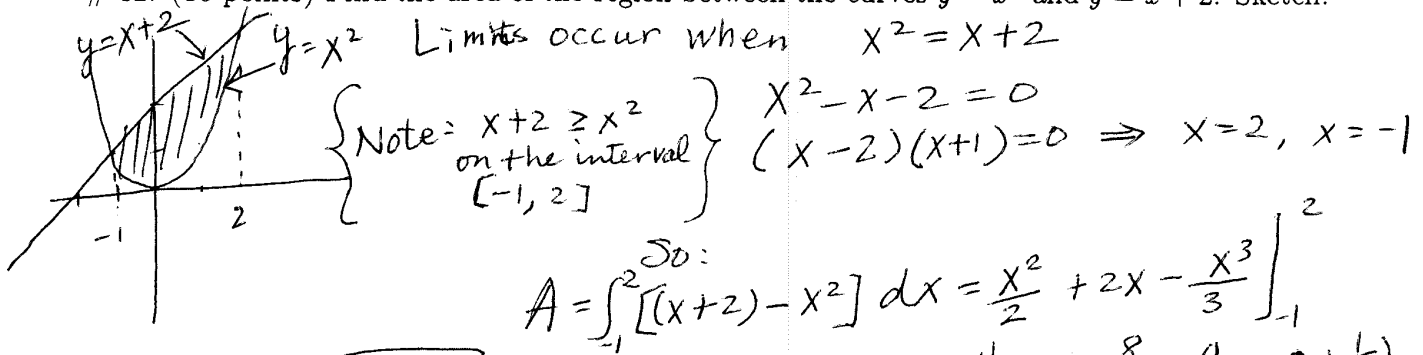
$$P(t) = 20,000 e^{-0.04621t}$$

Find $P(50)$:

$$P(50) = 20,000 e^{-0.04621(50)}$$

$$\approx 1984.23$$

12. (10 points) Find the area of the region between the curves $y = x^2$ and $y = x + 2$. Sketch.



Limits occur when $x^2 = x + 2$

Note: $x + 2 \geq x^2$ on the interval $[-1, 2]$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0 \Rightarrow x = 2, x = -1$$

So:

$$A = \int_{-1}^2 [(x+2) - x^2] dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3} \right)$$

$$= \frac{9}{2}$$

$A = \frac{9}{2}$

13. (10 points) Find the consumers' surplus for the demand curve $p = 30 - (0.2)x^2$ at the sales level $x = 10$.

where $A = 10$ $B = p(10) = 10$

$$\int_0^A (p(x) - B) dx$$

$$\int_0^{10} [(30 - 0.2x^2) - 10] dx = \int_0^{10} (20 - 0.2x^2) dx = \int_0^{10} (20 - \frac{1}{5}x^2) dx$$

$$= 20x - \frac{x^3}{15} \Big|_0^{10} = 200 - \frac{1000}{15} = \frac{2000}{15} = \frac{400}{3} \approx 133.\bar{3}$$

14. (15 points) Evaluate the integrals.

(a). $\int_1^2 (3x^2 + \frac{8}{x^3} + 1) dx = \int_1^2 (3x^2 + 8x^{-3} + 1) dx$

$$= \left[x^3 - \frac{4}{x^2} + x \right]_1^2 = [8 - 1 + 2] - [1 - 4 + 1] = 11$$

(b). $\int \frac{x^2}{x^3 + 4} dx$ let $u = x^3 + 4 \Rightarrow du = 3x^2 dx$ $\frac{1}{3} du = x^2 dx$

$$= \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln|u| + C = \frac{1}{3} \ln|x^3 + 4| + C$$

(c). $\int (4 + 10e^{-2x}) dx$

$$= 4x + \frac{10e^{-2x}}{-2} + C = 4x - 5e^{-2x} + C$$

- # 15. (15 points) Let $f(x, y) = 2x^4 + 4xy + y^2$. Find all possible relative maximum and relative minimum points of $f(x, y)$. Use the second derivative test to determine the nature of each such point. If the second derivative test is inconclusive, so state.

Recall: $D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$.

$$\frac{\partial f}{\partial x} = 8x^3 + 4y \rightarrow 8x^3 + 4y = 0 \rightarrow 4y = -8x^3 \rightarrow y = -2x^3$$

$$\frac{\partial f}{\partial y} = 4x + 2y \rightarrow 4x + 2y = 0 \rightarrow 2y = -4x \rightarrow y = -2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4$$

$$-2x^3 = -2x$$

$$x^3 = x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x(x+1)(x-1) = 0$$

$$x = 0, x = -1, x = 1$$

possible max/mins are:

$$(0, 0)$$

$$(-1, 2)$$

$$(1, -2)$$

$$D(x, y) = (24x^2)(2) - [4]^2$$

$$D(0, 0) < 0 \rightarrow \text{neither max nor min}$$

$$D(-1, 2) = 32 > 0 \text{ and } \frac{\partial^2 f}{\partial x^2} > 0 \text{ min}$$

$$D(1, -2) = 48 - 16 > 0 \text{ and } \frac{\partial^2 f}{\partial x^2} > 0 \text{ min}$$

- # 16. (10 points) Use Lagrange Multipliers to maximize $x^2 + xy + y^2 - 4x - 2y$ subject to the constraint $1 + x - y = 0$.

$$F(x, y, \lambda) = x^2 + xy + y^2 - 4x - 2y + \lambda(1 + x - y)$$

$$\frac{\partial F}{\partial x} = 2x + y - 4 + \lambda = 0$$

$$\frac{\partial F}{\partial y} = x + 2y - 2 - \lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = 1 + x - y = 0$$

$$4 - 2x - y = x + 2y - 2$$

$$6 - 3x = 3y$$

$$y = 2 - x$$

$$1 + x - (2 - x) = 0$$

$$2x - 1 = 0$$

$$x = \frac{1}{2} \quad \text{So } y = 2 - \frac{1}{2} = \frac{3}{2}$$

$\therefore \left(\frac{1}{2}, \frac{3}{2}\right)$ will maximize the function with the given restraint