Math 160 Final Exam
December 15, 1998

Name Key
Section Number 1
Instructor

Do all your writing on these six pages, using the backs if necessary. You must show your work, and simplify all answers, unless told otherwise.

\# 1. (10 points) Find an equation of the line through the points (3,-4) and (8,6).

$$M = \frac{6+4}{8-3} = \frac{10}{5}$$

$$m = 2$$

$$y+4 = 2(x-3)$$

$$y+4 = 2x-6$$

$$2x-y = 10$$

\# 2. (10 points) Find an equation of the tangent line to the curve $y = x^{3/2} - x^2 + 9$ at the point (4,1).

$$m = y'(4)$$

$$y' = \frac{3}{2}x^{\frac{1}{2}} - 2x$$

Eq: $y-1 = -5(x-4)$

$$y' = \frac{3\sqrt{x}}{2} - 2x$$

$$m = \frac{3\sqrt{4}}{2} - 2(4) = \frac{3(2)}{2} - 8 = 3-8 = \underline{-5}$$

\# 3. (10 points) Let $f(x) = 5x^2 - 3x + 2$. Use the limit definition of the derivative to show that $f'(x) = 10x - 3$.

$$f'(x) = \lim_{h \to 0} \frac{5(x+h)^2 - 3(x+h) + 2 - (5x^2 - 3x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{5(x^2 + 2xh + h^2) - 3x - 3h + 2 - 5x^2 + 3x - 2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{5x^2} + 10xh + 5h^2 \cancel{-3x} - 3h \cancel{+2} \cancel{-5x^2} \cancel{+3x} \cancel{-2}}{h}$$

$$= \lim_{h \to 0} \frac{10xh + 5h^2 - 3h}{h} = \lim_{h \to 0} \frac{\cancel{h}(10x + 5h - 3)}{\cancel{h}} = \underline{\underline{10x - 3}}$$

\# 4. (20 points) (a). Complete the formula: $\frac{d}{dx}(f(x)g(x)) =$ $f'(x)g(x) + f(x)g'(x)$

(b). Find $\lim_{x\to 3} \frac{x-3}{x^2-9}$. $= \lim_{x\to 3} \frac{\cancel{x-3}}{(\cancel{x-3})(x+3)} = \frac{1}{6}$

(c). Solve for x in $250 = 1000e^{-2.34x}$.

$\frac{1}{4} = e^{-2.34x}$ $\ln\left(\frac{1}{4}\right) = -2.34x$ $x = \frac{\ln(1/4)}{-2.34}$

(d). Solve for x in $2 + 3\ln(x+5) = 12$.

$3\ln(x+5) = 10$ $\ln(x+5) = 7$ $x+5 = e^7$ $x = e^7 - 5$

\# 5. (30 points) Find the derivative of each of the following. You need not simplify.

(a). $y = 4x^9 + 7x^4 - 4x^3 + x^2 + 100$.

$y' = 36x^8 + 28x^3 - 12x^2 + 2x$

(b). $y = (2x^2 + 3x - 1)^{10}$.

$y' = 10(2x^2+3x-1)^9 \cdot (4x+3)$

(c). $y = \frac{3x-8}{x^2+4}$.

$y' = \frac{(x^2+4)(3) - (3x-8)(2x)}{(x^2+4)^2}$

(d). $f(x) = x^2e^x$.

$f' = e^x \cdot 2x + x^2e^x$

(e). $y = e^{x^3-5x}$.

$y' = e^{x^3-5x} \cdot (3x^2-5)$

(f). $g(x) = \ln(2x^3+1)$.

$g' = \frac{1}{2x^3+1} \cdot 6x^2$

\# 6. (15 points) Using calculus, find all relative maximum and relative minimum points and all inflection points of $y = f(x) = 2x^3 - 3x^2 - 12x + 1$. Use this information to sketch the graph.

$f' = 6x^2 - 6x - 12$

$6(x^2 - x - 2) = 0$

$6(x-2)(x+1) = 0$

$x = 2,\ x = -1$

$f'' = 12x - 6$ $\quad f''(2) = >$ so $(2,-19)$ is a min

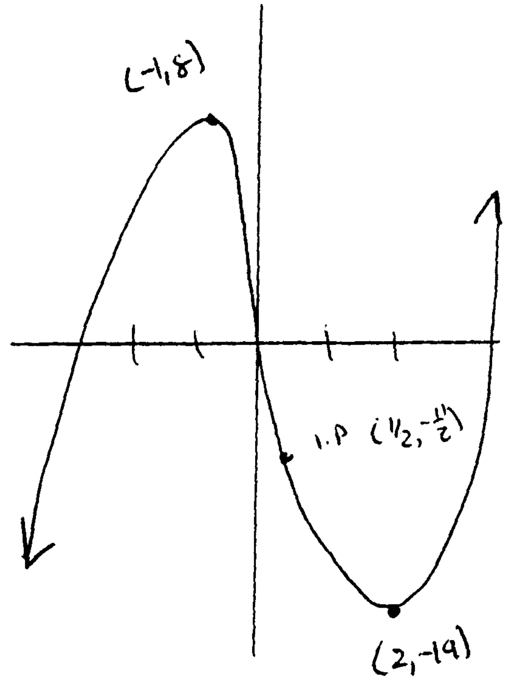
$f''(-1) = <$ so $(-1, 8)$ is a max

$12x - 6 = 0$

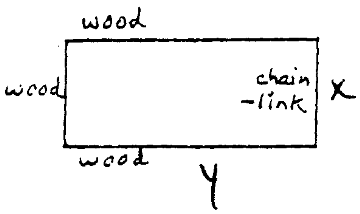
$6(2x-1) = 0$

$x = \frac{1}{2}$ is a possible inflection point

f'' $\quad$ - - - - 0 + + + + $\quad \frac{1}{2}$ $\quad$ $(\frac{1}{2}, \frac{-11}{2})$ is an inflection point



\# 7. (15 points) A rectangular pen with area 200 square feet is to be constructed. Three sides are to be made of wood fencing and the fourth side is to be made of chain-link fencing. Wood fencing costs \$ 2 per foot and chain-link fencing costs \$ 6 per foot. Find the dimensions of the cheapest such pen.



minimize cost

$C = 2y + 2x + 2y + 6x$

$C = 4y + 8x$ $\qquad xy = 200$ $\qquad y = \frac{200}{x}$

$C(x) = 4\left(\frac{200}{x}\right) + 8x$

$C(x) = \frac{800}{x} + 8x$ $\qquad C'(x) = -\frac{800}{x^2} + 8$ $\qquad 8 = \frac{800}{x^2}$

$8x^2 = 800$

$x^2 = 100$

$\boxed{x = 10}$

$\boxed{x = 10 \text{ so } y = 20}$

\# 8. (10 points) The demand equation for a certain commodity is $p = 200 - 3x$, where x is the number of units and p is the price per unit in dollars. The cost function is $C(x) = 5450 + 2x$. What number of units and price per unit result in the maximum profit?

$P = R - C$ where $R = xp$

$P(x) = x(200-3x) - [5450 + 2x]$

$P(x) = 200x - 3x^2 - 5450 - 2x$

$P(x) = 198x - 3x^2 - 5450$

$P' = 198 - 6x$ $\quad (P'=0)$: $198 - 6x = 0$ $\quad 198 = 6x$ $\quad \boxed{x = 33}$ $\quad$ so $p = 200 - 3(33)$ $\quad \boxed{p = \$101}$

\# 9. (10 points) The function $f(x) = xe^{-2x}$ has one relative maximum and no other extreme values. Find the value of x at which the maximum occurs.

$f' = e^{-2x} + xe^{-2x} \cdot -2$

$f' = e^{-2x} - 2xe^{-2x}$

$e^{-2x} - 2xe^{-2x} = 0$

$e^{-2x}[1 - 2x] = 0$ so $\boxed{x = \frac{1}{2}}$

\# 10. (10 points) A population is assumed to grow at a rate proportional to its size. At a starting date the population is 5.4 million, and seven months later the population is 9.6 million. What will be the population 25 months after the starting date? Give a decimal answer.

$P(t) = P_0 e^{kt}$ $\quad P(t) = 5.4e^{kt}$ $\quad$ Find k first

note: $P(7) = 9.6$ so $9.6 = 5.4e^{7k}$ $\quad \frac{9.6}{5.4} = e^{7k}$ $\quad \ln\left(\frac{9.6}{5.4}\right) = 7k$

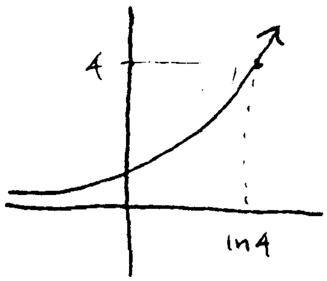
$k = \frac{\ln\left(\frac{9.6}{5.4}\right)}{7}$

$\approx .0822$

$P(t) = 5.4e^{.0822t}$

Find $P(25) = 5.4e^{.0822(25)}$ $\approx$ 42.16 million

\# 11. (10 points) Find the area of the region between the curves $y = e^x$ and $y = 4$ from $x = 0$ to $x = \ln 4$. Sketch.



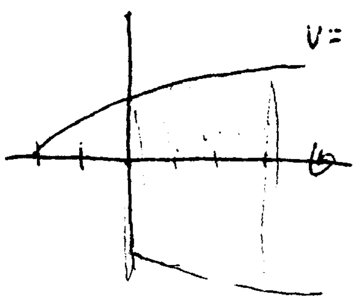
$$\int_0^{\ln 4} (4 - e^x) = 4x - e^x \Big|_0^{\ln 4}$$

$$= 4\ln 4 - e^{\ln 4} - (0 - e^0)$$

$$= \ln 4^4 - 4 - (-1)$$

$$\ln 256 - 4 + 1 = \underline{\ln 256 - 3}$$

\# 12. (10 points) Find the volume of the solid of revolution generated by revolving around the x-axis the following region: the area below $y = \sqrt{x+2}$ from $x = 0$ to $x = 3$.



$$V = \int_0^3 \pi(\sqrt{x+2})^2 dx = \pi \int_0^3 (x+2) dx = \pi\left(\tfrac{1}{2}x^2 + 2x\right)\Big|_0^3$$

$$= \pi\left(\tfrac{1}{2}(9) + 6\right) = \pi\left(\tfrac{9}{2} + 6\right)$$

$$\pi\left(\tfrac{21}{2}\right) = \frac{21\pi}{2}$$

\# 13. (15 points) Evaluate the integrals.

(a). $\int \left(\frac{1}{x^2} + 2\right) dx.$ $\quad = \int (x^{-2} + 2)\, dx = \frac{1}{-1} x^{-1} + 2x + C$

(b). $\int \frac{e^x}{5 + e^x} dx.$ $\quad u = 5 + e^x \quad du = e^x dx$

$$\int \frac{1}{u} du = \ln|u| + C = \ln|5 + e^x| + C$$

(c). $\int 4e^{-3x} dx.$ $\quad u = -3x \quad du = -3dx$

$$-\frac{4}{3}\int e^u du = -\frac{4}{3}e^u + C = \underline{-\frac{4}{3}e^{-3x} + C}$$

14. (15 points) Let $f(x,y) = \frac{1}{5}x^5 + x^2y - \frac{1}{2}y^2$. Find all possible relative maximum and relative minimum points of $f(x,y)$. Use the second derivative test to determine the nature of each such point. If the second derivative test is inconclusive, so state.

Recall: $D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$.

$\frac{df}{dx} = x^4 + 2xy$, $\frac{df}{dy} = x^2 - y$ max or min when $\frac{df}{dx}$, $\frac{df}{dy}$ equal 0

$x^4 + 2xy = 0$

$x^2 - y = 0$ so $\underline{y = x^2}$

$x^4 + 2x(x^2) = 0$

$x^4 + 2x^3 = 0$

$x^3(x+2) = 0$

$x=0$, $x=-2$ possible extrema at $\underline{(0,0) \text{ and } (-2,4)}$

$D(x,y) = (4x^3 + 2y)(-1) - (2x)^2$

$D(0,0) = 0$ so $\underline{\text{Inconclusive}}$

$D(-2,4) = (-32+8)(-1) - (-2)^2$

$= (-24)(-1) - 4 = 20 > 0$

and $\frac{d^2f}{dx^2}(-2,4) < 0$ so

m @ $(-2,4)$

+ Inconclusive @ $(0,0)$

15. (10 points) Use Lagrange Multipliers to find the maximum value of $f(x,y) = 2 - x^2 - y^2$ subject to the constraint $y + 2x + 1 = 0$.

$F(x,y,\lambda) = 2 - x^2 - y^2 + (y + 2x + 1)\lambda$ so $F(x,y,\lambda) = 2 - x^2 - y^2 + y\lambda + 2x\lambda + \lambda$

$\frac{dF}{dx} = -2x + 2\lambda = 0$ $2\lambda = 2x$ $\underline{\lambda = x}$

$F_y = -2y + \lambda = 0$ $\underline{\lambda = 2y}$ so $x = 2y$

$F_\lambda = y + 2x + 1 = 0$

$y + 2(2y) + 1 = 0$

$y + 4y + 1 = 0$

$5y = -1$

$y = \frac{-1}{5}$ so $x = \frac{-2}{5}$ Therefore the max value is

$f(-2/5, -1/5) = 2 - (\frac{-1}{5})^2 - (\frac{-2}{5})^2 = 2 - \frac{1}{25} - \frac{4}{25} = \frac{45}{25}$