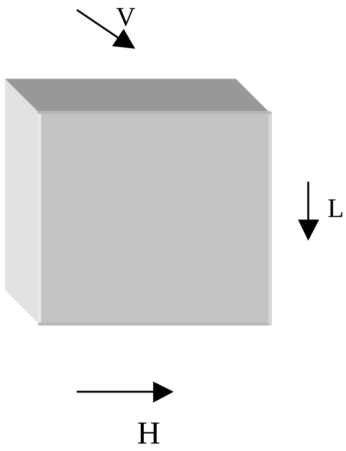
Group of Motions of a Cube



<u>Face numbers</u>: Front=1, Right=2, Back=3, Left=4, Up=5, Down=6

Motion	**Permutation of faces in cycle notation**
Identity (no motion)	(1)
H	(1234)
H^2	(13)(24)
H^3	(1432)
V	(1536)
HV	(154)(236)
H^2V	(15)(24)(36)
H^3V	(152)(364)
V^2	(13)(56)
$H\ V^2$	(14)(23)(56)
$H^2\ V^2$	(24)(56)
$H^3\ V^2$	(12)(34)(56)
V^3	(1635)
$H\ V^3$	(164)(235)
$H^2\ V^3$	(16)(24)(35)
$H^3\ V^3$	(162)(354)
L	(2645)
HL	(126)(345)
$H^2\ L$	(13)(26)(45)
$H^3\ L$	(145)(263)
L^3	(2546)
HL^3	(125)(346)
H^2L^3	(13)(25)(46)
H^3L^3	(146)(253)

Problem: How many distinct ways can we color the faces of a cube, using one of three colors for each face?

We view the group of motions, G, of the cube as permutations of its six faces. Thus G can be considered as a subgroup of S_6. There are 24 elements in this group. Actually G is isomorphic to S_4, since it can also be considered as the set of all permutations of the 4 diagonals that connect the four pairs of opposite corners of the cube.

In the table above, there are 6 four-cycles of the type (1234); 6 products of 3 transpositions of the type (12)(34)(56); 3 products of two transpositions of the type (12)(34); and 8 products of two three-cycles of the type (123)(456). Using Burnside's formula, we have:

$$\frac{1}{|G|}\sum_{g\in G}|X_g| = \frac{1}{24}(3^6 + 12\times 3^3 + 3\times 3^4 + 8\times 3^2) = \frac{1368}{24} = 57$$

Thus there are 57 distinct ways we can color the faces of a cube, using one of three colors for each face.